SURFACE WAVE PROPAGATION ALONG FERRITE-COATED CYLINDRICAL CONDUCTOR

by

John Aagesen

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Department of Electrical Engineering

University of Toronto

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SUMMARY - The paper deals with a theoretical investigation of surface waves propagating along a ferrite-coated, perfectly conducting cylinder. An approximate expression determining the propagation constant is derived, and a brief numerical analysis is carried out in order to give examples of the variation of the phase velocity with the involved parameters and to establish the order of allowed dimensions if the approximations made should be justified.

1. INTRODUCTION.

During the last ten years considerable interest has been devoted to surface wave transmission lines and antennas. The term "surface wave" is here used in the sense: An electromagnetic wave that propagates without radiation along the interface between two different media.

The concept of the existence of such waves is not new. In 1899 Sommerfeld[1] showed that a wave can propagate without radiation along an infinitely long cylindrical wire of finite conductivity. In 1907 Zenneck[2] described a wave that travels along a plane boundary between two homogeneous media of different conductivity and permittivity.

Although the surface wave theory is not new most of the work has been done in recent years. The most significant step forward was taken by Goubau[3] who in 1950 published a paper in which he deals with the propagation of axially symmetric waves along the surface of a perfectly conducting cylinder. If the surface of the cylinder was not modified in some way no such surface waves would exist. However, by applying

a thin coating of a dielectric to the surface, or by corrugating this, Goubau shows that the transmission system can support an axially symmetric TM mode. Higher modes have been investigated for the Sommerfeld case by Hondros[4], but they can be shown to be of no practical interest because most of the propagated energy is confined to the interior of the conductor. The purpose of the modification of the surface is to bring down the phase velocity (which for a non-leaking surface wave must be less than the free-space velocity in the surrounding medium) in the object of increasing the decay of the fields in the transverse direction. This means further that the size of the launching device may be reduced.

In most of the literature on this subject the coating, that has been applied to the conducting surface for above-mentioned purpose, has been isotropic, namely a dielectric. The present paper considers a perfectly conducting cylinder coated by a thin layer of ferrite which is DC magnetized in the longitudinal direction. The characteristic aspect of such a system is that the propagation constant of a surface wave propagating in the axial direction can be changed by varying the DC magnetization. This suggests that the structure may, for example, be applied as a surface wave antenna with the interesting property that the radiation pattern can be controlled electrically by adjusting the current through the magnetizing solenoide.

In 1958 Pease[5] published an article treating surface waves traveling over a ferrite-coated conducting plane. He

obtains expressions for the propagation constant of the wave propagating over the surface for three typical directions of the DC magnetization, one of these being coinciding with the direction of propagation. He does not carry out any numerical analysis and his discussion, therefore, reveals nothing about the allowed order of thickness of the ferrite film in practical problems.

The following analysis contains Goubau´s as well as Pease´s results as special cases. The method used is based on the approach of Suhl and Walker[6]; a secular equation from which the propagation constant can be found by a semigraphical method is derived under the condition of large radius of the conductor and small thickness of the ferrite coating. At the end of the paper the numerical results of some specific examples are given.

2. SYMBOLS.

The following list contains the symbols which are used frequently throughout the text.

a, radius of the cylindric conductor.

δ, thickness of the ferrite coating.

$a_o = a + \delta$, outer radius of the ferrite-coated structure.

r, φ, z, cylindrical coordinate system.

$\bar{E}$, electric field vector.

$\bar{H}$, magnetic field vector.

Field components in directions of coordinate axes indicated by E or H with proper subscripts.

C and D constants determining the magnitudes of the z-components of the electric and magnetic fields, respectively, for the surface wave.

ω, angular frequency.

k_1, constant determining transverse variation of surface wave.

β, phase constant of surface wave.

χ, constant determining transverse variation of fields within ferrite.

n, mode number.

p, Hankel function $H_n^{(1)}(k_1 r)$.

p', $H_n^{(1)'}(k_1 r)$.

ε, dielectric constant of ferrite.

ε_o, dielectric constant of vacuum.

μ_o, permeability of vacuum.

$[\mu] = \begin{bmatrix} \mu & -j\varkappa & 0 \\ j\varkappa & \mu & 0 \\ 0 & 0 & \mu_o \end{bmatrix}$, tensor permeability of magnetized ferrite.

$\nu = \frac{\mu}{\mu_o}$.

$\varsigma = \frac{\varkappa}{\mu}$.

$q = 1 - \frac{\beta^2}{\nu}$.

H_o, steady magnetic field within ferrite.

M_o, saturation magnetization of ferrite.

$|\gamma| = 0.0352 \frac{\text{Mc/s}}{\text{A/m}}$, gyromagnetic ratio.

k_o, free-space propagation constant of vacuum.

The rationalized MKS unit system is used throughout and the time factor is $e^{j\omega t}$.

3. GENERAL SOLUTION OF MAXWELL'S EQUATIONS IN THE TWO MEDIA.

The z-dependence of the fields is assumed to be given by $e^{-j\beta z}$ and the ferrite is magnetized by a steady magnetic field in the z-direction. Fig. 1 shows the situation of the ferrite-coated, perfectly conducting cylinder in the cylindrical coordinate system r,φ,z; the axis of the cylinder coincides with the z-axis. Radius of the conductor is a, and the layer of ferrite applied to the surface of this has the thickness δ.

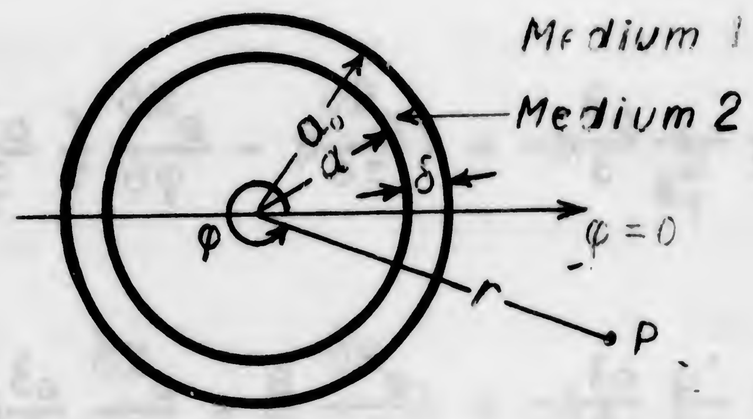

Fig. 1. Cross-section of the investigated cylindrical structure and its position in coordinate system.

3.1. Fields in the Surrounding Medium.

The solution of Maxwell's equations in the isotropic Medium 1, which is assumed to be vacuum, is well known. With the adopted normalizing convention, according to which all

lengths are multiplied with $\omega\sqrt{\varepsilon\mu_o}$ and magnetic field strengths are multiplied by $\sqrt{\frac{\mu_o}{\varepsilon}}$, the field components in Medium 1 are

(1a) $$E_z = CH_n^{(1)}(k_1 r) = Cp$$

(1b) $$H_z = DH_n^{(1)}(k_1 r) = Dp$$

(1c) $$E_r = -\frac{1}{k_1^2}\left(\beta\frac{\partial E_z}{\partial r} + \frac{j}{r}\frac{\partial H_z}{\partial \varphi}\right) = -\beta\frac{p'}{k_1}C - n\frac{p}{k_1^2}\frac{1}{r}D$$

(1d) $$E_\varphi = \frac{1}{k_1^2}\left(-\frac{\beta}{r}\frac{\partial E_z}{\partial \varphi} + j\frac{\partial H_z}{\partial r}\right) = -jn\beta\frac{p}{k_1^2}\frac{1}{r}C + j\frac{p'}{k_1}D$$

(1e) $$H_r = \frac{1}{k_1^2}\left(j\frac{\varepsilon_o}{\varepsilon}\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \beta\frac{\partial H_z}{\partial r}\right) = -n\frac{\varepsilon_o}{\varepsilon}\frac{p}{k_1^2}\frac{1}{r}C - \beta\frac{p'}{k_1}D$$

(1f) $$H_\varphi = -\frac{1}{k_1^2}\left(j\frac{\varepsilon_o}{\varepsilon}\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \varphi}\right) = -j\frac{\varepsilon_o}{\varepsilon}\frac{p'}{k_1}C + jn\beta\frac{p}{k_1^2}\frac{1}{r}D$$

$$p = H_n^{(1)}(k_1 r);\quad p' = H_n^{(1)\prime}(k_1 r).$$

$$k_1^2 = \frac{\varepsilon_o}{\varepsilon} - \beta^2$$

The usefulness of the normalization is evident from the analysis of Appendix I.

Since it is the object of the present paper to investigate non-leaking surface waves, the fields in a transverse plane must decrease in a static manner with increasing r. This

means that β must be greater than the free-space propagation constant in Medium 1 and the solution for k_1 to be used is

$$k_1 = j\sqrt{\beta^2 - \frac{\varepsilon_o}{\varepsilon}}$$

In order to obtain decaying fields in the transverse direction only the Hankel function of first kind can be used in the expressions for E_z and H_z. If k_1 had been chosen equal to $-j\sqrt{\beta^2 - \varepsilon_o/\varepsilon}$, the Hankel function of second kind, exclusively, would have been involved.

In the field expressions a factor of $e^{j(n\varphi + \omega t - \beta z)}$, n any integer, is omitted. That the φ-variation is given by the factor $e^{jn\varphi}$ is due to the fact that the solution for the fields in the ferrite requires such a φ-dependence. Thus by the continuity of the tangential components of the fields through the boundary between the two media the same factor must express the φ-variation of the fields in Medium 1.

In all the following field expressions the dependence of any component upon φ, t and z is of the form $e^{j(n\varphi + \omega t - \beta z)}$, but this factor is suppressed for simplicity.

3.2. Fields in the Ferrite.

The derivation of the general field expressions in a ferrite medium, magnetized in the direction of propagation of the fields, has been treated by Suhl and Walker[6] and the present analysis has been based on their approach. In the analysis anisotropy and exchange fields have been neglected, so that the total magnetic field in the ferrite consists of the DC

field, H_o, and the AC field, $\bar{H}$, this being very small compared with H_o. The assumption of negligible effect of the anisotropy holds for sufficiently high frequencies ($f > \sim 3000$ Mc/s for $H_o \sim 10^5$ A/m; lower frequency limit for higher magnetizations). When the ferrite is not very close to gyromagnetic resonance, the exchange fields as well as losses may be neglected.

The tensor permeability for the ferrite is

$$[\mu] = \begin{bmatrix} \mu & -j\varkappa & 0 \\ j\varkappa & \mu & 0 \\ 0 & 0 & \mu_o \end{bmatrix} \tag{2}$$

assuming that the steady magnetic field is so strong that the ferrite is fully saturated.

The derivation of the general field expressions in the ferrite by the method of Suhl and Walker is reviewed in a slightly modified form in Appendix I.

The field components E_z, H_z, E_φ and H_φ in the ferrite as derived in Appendix I, equations (37a), (37b), (42a) and (42b), are as follows

$$E_z = \frac{\Lambda_2\Psi_1 - \Lambda_1\Psi_2}{\Lambda_2 - \Lambda_1} \tag{3a}$$

$$H_z = j\frac{\Psi_1 - \Psi_2}{\Lambda_2 - \Lambda_1} \tag{3b}$$

$$\nu(q^2 - \rho^2)(\Lambda_1 - \Lambda_2)E_\varphi = -\frac{n\beta}{r}\left[q(\Lambda_2\Psi_1 - \Lambda_1\Psi_2) + \beta\rho(\Psi_1 - \Psi_2)\right] + \frac{\partial}{\partial r}\left[\beta\rho(\Lambda_2\Psi_1 - \Lambda_1\Psi_2) + \nu(q - \rho^2)(\Psi_1 - \Psi_2)\right] \tag{3c}$$

$$(3d)\quad j\nu(a^2-\rho^2)(\Lambda_1-\Lambda_2)H_\phi = \frac{n}{r}\left[\rho(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta a(\psi_1-\psi_2)\right] - \frac{\partial}{\partial r}\left[a(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta\rho(\psi_1-\psi_2)\right]$$

where ψ_1 and ψ_2 are given by

$$(4a)\qquad \psi_1 = A_1' H_n^{(1)}(\chi_1 r) + B_1' H_n^{(2)}(\chi_1 r)$$

$$(4b)\qquad \psi_2 = A_2' H_n^{(1)}(\chi_2 r) + B_2' H_n^{(2)}(\chi_2 r)$$

and Λ_1, Λ_2 , χ_1 and χ_2 are determined by equations (32) and (33).

4. DERIVATION OF SIMULTANEOUS EQUATIONS DETERMINING PROPAGATION CONSTANT AND FIELD COMPONENTS.

The next step toward the solution of the given problem will be to derive a pair of simultaneous equations that determines β and the mutual magnitude of the z-components of the magnetic and electric fields.

The boundary conditions require that the tangential component of the electric field vanishes at the surface of the perfectly conducting cylinder, and that the tangential components of the electric and magnetic fields be continuous through the boundary between Media 1 and 2. This can be expressed by

$$(5a) \qquad \left[E_z\right]_{r=a} = 0$$

$$(5b) \qquad \left[E_\varphi\right]_{r=a} = 0$$

$$(6a) \qquad \left[E_z\right]_{r=a+\delta-} = \left[E_z\right]_{r=a+\delta+}$$

$$(6b) \qquad \left[E_\varphi\right]_{r=a+\delta-} = \left[E_\varphi\right]_{r=a+\delta+}$$

$$(6c) \qquad \left[H_z\right]_{r=a+\delta-} = \left[H_z\right]_{r=a+\delta+}$$

$$(6d) \qquad \left[H_\varphi\right]_{r=a+\delta-} = \left[H_\varphi\right]_{r=a+\delta+}$$

From (3c) it is seen that if the φ-dependence had been chosen completely general as $e^{jn\varphi} + Ke^{-jn\varphi}$, the left-hand side of (5b) would have been a linear combination of the form

$$K_1(e^{jn\varphi} - Ke^{-jn\varphi}) + K_2(e^{jn\varphi} + Ke^{-jn\varphi})$$

$$= (K_1 + K_2)e^{jn\varphi} - (K_1 - K_2)Ke^{-jn\varphi}$$

where K_1 and K_2 are parameters depending upon the properties of the entire system. For $n \neq 0$, $K_1 \neq 0$ as seen from equations (3a), (3b), (3c), (5a) and (5b). Hence above expression cannot identically equal zero unless $K_1 + K_2 = (K_1 - K_2)K = 0$, which is only fulfilled for $K = 0$. For $n = 0$ the applicability of the factor $e^{jn\varphi}$ is obvious.

A rigorous solution of the problem turns out to be extremely complex, two simplifying assumptions will therefore be introduced. These are

$$(7) \qquad \begin{aligned} |\chi|a &\gg 1 \\ |\chi|\delta &\ll 1 \end{aligned}$$

The first condition has the effect that the Hankel functions involved in the expressions for the ψ's may be substituted by their asymptotic values. Hence equations (4a) and (4b) may be written

$$\psi_1 \doteq A_1 \frac{e^{j\chi_1 r}}{\sqrt{r}} + B_1 \frac{e^{-j\chi_1 r}}{\sqrt{r}} \tag{8a}$$

$$\psi_2 \doteq A_2 \frac{e^{j\chi_2 r}}{\sqrt{r}} + B_2 \frac{e^{-j\chi_2 r}}{\sqrt{r}} \tag{8b}$$

The second condition $|\chi|\delta \ll 1$ justifies the approximation that $e^{j\chi\delta}$ can be substituted by the two first terms of its Taylor series. Thus

$$e^{\pm j\chi\delta} \doteq 1 \pm j\chi\delta \tag{9}$$

Under the restricting conditions of (7) the field components in the two media as given in equations (1a) to (1f) and (3a) to (3d), when substituted into the boundary conditions (5a) to (6d), results in two simplified simultaneous equations. The detailed mathematical analysis is carried out in Appendix II. The magnitude of the field components of the surface wave and the propagation constant can be derived form the resultant pair of simultaneous equations (56) and (58) of the Appendix II. These equations are quoted below.

$$(10a)\quad \beta\left[\frac{\varsigma\delta(a+\delta)-\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}+n(1+\frac{\delta}{2a})+n^2\frac{1-q}{q^2-\varsigma^2}q\varsigma\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}+jn\nu\frac{1}{k_1^2}\right]C$$

$$-j\nu\left[\frac{(q-\varsigma^2)\left[\delta(a+\delta)-\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}\right]+n^2\frac{(1-q)^2}{q^2-\varsigma^2}\varsigma^2\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}+\frac{1}{k_1}\frac{p'}{p}(a+\delta)\right]D=0$$

$$(10b)\quad \left[\frac{\frac{1}{\delta}(a+\delta)+q\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}-n\varsigma(1+\frac{\delta}{2a})-n^2\frac{(1-q)\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}-\frac{\nu}{k_1}\frac{\varepsilon_o}{\varepsilon}\frac{p'}{p}(a+\delta)\right]C$$

$$-j\beta\left[\frac{\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\xi}{4a}-n(1+\frac{\delta}{2a})-n^2q(1-q)\varsigma\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}-n\nu\frac{1}{k_1^2}\right]D=0$$

where $r=a+\delta$ is inserted in the expressions for p and p´.

These equations can have solutions for C and D which are not simultaneously zero only if the determinant of the system is zero. This leads to a secular equation for β which is not shown here. The secular equation is, however, extremely complex and can only be attacked by numerical means. When β is determined the ratio D/C can be found from either of the equations (10a) and (10b).

5. ANALYSIS OF AXIALLY SYMMETRIC SURFACE WAVE MODE.

A general solution of the given problem is, as mentioned, difficult due to the complexity of the final pair of simultaneous equations. Since the situation of most practical interest is the one corresponding to $n = 0$, i.e. axial symmetry of the fields, the further analysis will be confined to this case.

For $n = 0$ the characteristic set of equations can be written

$$(11a)\quad \beta\varrho\delta\left[1 - \frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{1}{4a(a+\delta)}\right]C - j\nu(q-\varrho^2)\left[\delta\left(1 - \frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{1}{4a(a+\delta)}\right) + \frac{1}{k_1}\,\frac{p'}{p}\left(1+\frac{\delta}{2a}\right)\right]D = 0$$

$$(11b)\quad \left[1 + q\,\frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{\delta^2}{4a(a+\delta)} - (q-\varrho^2)\frac{\varepsilon_o}{\varepsilon}\,\frac{\nu}{k_1}\,\frac{p'}{p}\,\delta\left(1+\frac{\delta}{2a}\right)\right]C - j\beta\varrho\,\frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{\delta^2}{4a(a+\delta)}\,D = 0$$

Equating the determinant of these equations to zero yields following quadratic equation in $k_1\frac{p}{p}$

$$(12)\qquad \left[1 - \frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{1}{4a(a+\delta)}\right]\delta\,\frac{1 + \frac{\delta^2}{4a(a+\delta)}}{1+\frac{\delta}{2a}}\left(k_1\frac{p}{p}\right)^2$$

$$+\left[1+(q+\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon})\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta^2}{4a(a+\delta)}-\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\delta^2\right]k_{1p}\frac{p}{}$$

$$-\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\delta(1+\frac{\delta}{2a})=0$$

An approximate solution for

$$k_{1p}\frac{p}{}=j\sqrt{\beta^2-\frac{\varepsilon_o}{\varepsilon}}\,\frac{H_o^{(1)}(j\sqrt{\beta^2-\frac{\varepsilon_o}{\varepsilon}}\,(a+\delta))}{H_o^{(1)\prime}(j\sqrt{\beta^2-\frac{\varepsilon_o}{\varepsilon}}\,(a+\delta))}$$

will now be found under the assumption that

$$(q+\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon})\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta^2}{4a(a+\delta)}-\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\delta^2$$

is small compared with unity. This is fulfilled as long as the propagation constant of the surface wave is only slightly greater than the free-space propagation constant for Medium 1. Since the limit of β for $\delta\rightarrow 0$ is equal to k_o, it is seen that above condition is valid for δ being sufficiently small. Further, since $\frac{\delta}{a}\ll 1$, $1+\frac{\delta^2}{4a(a+\delta)}\doteq 1$. Hence the discriminant of equation (12) can be written

$$\Delta\doteq 1+2\left[(q+\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon})\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta^2}{4a(a+\delta)}-\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\delta^2\right]$$

$$-4\left[\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\frac{q-\varsigma^2}{\varsigma^2-b}\frac{\delta^2}{4a(a+\delta)}-\nu(q-\varsigma^2)\frac{\varepsilon_o}{\varepsilon}\delta^2\right]+$$

$$= 1 + 2\left[(q - \nu(q-\varrho^2)\frac{\varepsilon_o}{\varepsilon})\frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{\delta^2}{4a(a+\delta)} + \nu(q-\varrho^2)\frac{\varepsilon_o}{\varepsilon}\,\delta^2\right]$$

$$\sqrt{\Delta} \doteq 1 + (q - \nu(q-\varrho^2)\frac{\varepsilon_o}{\varepsilon})\frac{q-\varrho^2}{q^2-\varrho^2}\,\frac{\delta^2}{4a(a+\delta)} + \nu(q-\varrho^2)\frac{\varepsilon_o}{\varepsilon}\,\delta^2$$

Of the two solutions obtained one can be eliminated, because it becomes infinite for $\delta = 0$, and since β is known to be $\frac{\varepsilon_o}{\varepsilon}$ in this case, $k_1\frac{p}{p'}$ must be equal to zero. Thus

$$k_1\frac{p}{p'} \doteq \nu(q-\varrho^2)\,\frac{\varepsilon_o}{\varepsilon}\,\delta\,(1+\frac{\delta}{2a}) \tag{13}$$

or, inserting the expressions for k_1, p, p´and q,

$$\sqrt{(\frac{\beta}{k_o})^2 - 1}\;\frac{jH_o^{(1)}\left[j\sqrt{(\frac{\beta}{k_o})^2 - 1}\,k_o(a+\delta)\right]}{-H_1^{(1)}\left[j\sqrt{(\frac{\beta}{k_o})^2 - 1}\,k_o(a+\delta)\right]} \doteq \left[\nu(1-\varrho^2)-\beta^2\right]k_o\delta(1+\frac{\delta}{2a}) \tag{14}$$

This has a real solution for β only in the case $\nu(1-\varrho^2) \geq \frac{\varepsilon_o}{\varepsilon}$.
It is convenient to introduce an approximation which is justified when β, for δ sufficiently small, can be substituted by $\frac{\varepsilon_o}{\varepsilon}$ on the right-hand side of (14). When this is done, it is noticed that (14) is valid also if the parameters k_o, β, a and δ are substituted by their actual, non-normalized values.

It is further practical to introduce the outer radius of the structure, a_o, as a parameter instead of a. Thus (14)

obtains the form

$$(15)\quad \sqrt{\left(\frac{\beta}{k_o}\right)^2-1}\,\frac{jH_o^{(1)}\left[j\sqrt{\left(\frac{\beta}{k_o}\right)^2-1}\;k_o a_o\right]}{-H_1^{(1)}\left[j\sqrt{\left(\frac{\beta}{k_o}\right)^2-1}\;k_o a_o\right]} \doteq \left[\nu(1-\varsigma^2)-\frac{\varepsilon_o}{\varepsilon}\right] k_o\delta\left(1+\frac{\delta}{2(a_o-\delta)}\right)$$

Two special cases have been treated by other authors. If the coating had been an ordinary, isotropic dielectric, $\nu(1-\varsigma^2)$ becomes unity and equation (14) is seen to agree with the expression found by Goubau[3].

When a_o tends to infinity (15) can be written

$$(16)\qquad \sqrt{\left(\frac{\beta}{k_o}\right)^2-1} \doteq \left[\nu(1-\varsigma^2)-\frac{\varepsilon_o}{\varepsilon}\right] k_o\delta$$

By a few rewritings including insertion of $\nu=\frac{\mu}{\mu_o}$ and $\varsigma=\frac{\varkappa}{\mu}$ this can be brought into a form identical to the expression obtained by Pease[5].

Insertion of the expression (13) into (11a) yields, with the assumptions made about the parameters, following approximate expression for $\frac{D}{C}$

$$\frac{D}{C} \doteq -j\beta\varsigma\frac{\varepsilon_o}{\varepsilon}\delta^2$$

For all situations that lead to a real solution for β, this expression is very small. This shows that the surface wave is mainly a TM mode, but a small TE component must be present

in order to satisfy Maxwell's equations. In the case of an isotropic dielectric coating $\varrho = 0$ and a pure TM wave is now supported by the structure.

When (13) is inserted into (11b) this equation becomes

$$q\frac{q-\varrho^2}{q^2-\varrho^2}\frac{\delta^2}{4a(a+\delta)} - j\beta\varrho\frac{q-\varrho^2}{q^2-\varrho^2}\frac{\delta^2}{4a(a+\delta)}\frac{D}{C} = 0$$

The second term can generally be neglected in comparison with the first one, and the equation is approximately satisfied because of the small value of the left-hand side (second order in $\frac{\delta}{a}$).

6. NUMERICAL RESULTS.

In the following the results of a brief numerical analysis are given in graphical form. Tables of the functions $jH_o^{(1)}(jy)$ and $-H_1^{(1)}(jy)$ are available[7]. The calculations are carried out for a specific material like Ferramic R-1 or R-3 with saturation magnetization 0.3 Wb/m^2,[8] and relative dielectric constant 10.

The graphs in figs. 2 and 3 are useful in solving equation (15).

The analysis shows that the phase velocity v (fig. 4) increases with increasing DC magnetization. This is due to the fact that $\sigma = \frac{|\gamma| H_o}{\omega}$ increases with H_o and, as seen from

fig. 2, $\nu(1-\varrho^2)$ will decrease when σ increases. Hence the right-hand side of (15) decreases, when H_o increases, which by comparison with fig. 3 shows that $\frac{\beta}{k_o}$ will decrease.

For constant magnetization $\frac{v}{c}$ decreases with frequency similarly to the case of dielectric coating, only the effect of frequency variation is more pronounced when the frequency approaches gyromagnetic resonance (fig. 5). The behavior of $\frac{v}{c}$, when the frequency increases, is the combined result of the variation of the involved parameters. σ and p will decrease, $\nu(1-\varrho^2)$ will therefore increase. The "electrical thickness" of the ferrite coating will increase. These phenomena will tend to increase $\frac{\beta}{k_o}$. Mentioned effects will, however, be counteracted by the increasing "electrical diameter" of the entire system. The effect of the latter variation is not as significant, though, as the one of the first two, hence the resultant effect is the shown drop in $\frac{v}{c}$ with increasing frequency. The continuation of $\frac{v}{c}$ above the line $\frac{v}{c} = 1$ is uncertain because the propagation constant becomes complex and the solution of equation (14), with $\beta - j\alpha$ substituted for β will be difficult and beyond the scope of this paper. Furthermore, the frequency will approach the resonance region where the entire theory will not hold because of the simplifying assumptions about the ferrite. However, a possible variation is intimated by the dotted continuations of the corresponding curves.

Finally, from fig. 6, it is seen that by letting the DC magnetic field vary with frequency, falling as well

as rising phase velocity curves may be obtained.

For small deviations of β from k_o, as considered here, (36) can with approximation be written

$$\lambda^2 - (p + 0.9\sigma)\lambda - 0.1 = 0 \tag{17}$$

(normalized $\beta^2 \doteq 0.1$). From this equation and (35) $|\chi|$ can be found and the justification of the approximations may be verified. An analysis shows that for the values of $\frac{v}{c}$ appearing in the graphs $|\chi|$ a $> \sim 3$, so that the substitution of the Hankel functions of this argument with their asymptotic values yields a good approximation. $|\chi|\delta$ does not exceed 0.8, a rather high value which it, however, only obtains at the highest values of δ considered; it is noted that χ, a and δ are here the normalized values so that a and δ increases with frequency. Hence only at the highest frequencies approaches $|\chi|\delta$ 0.8, and the approximation involved here is still acceptable.

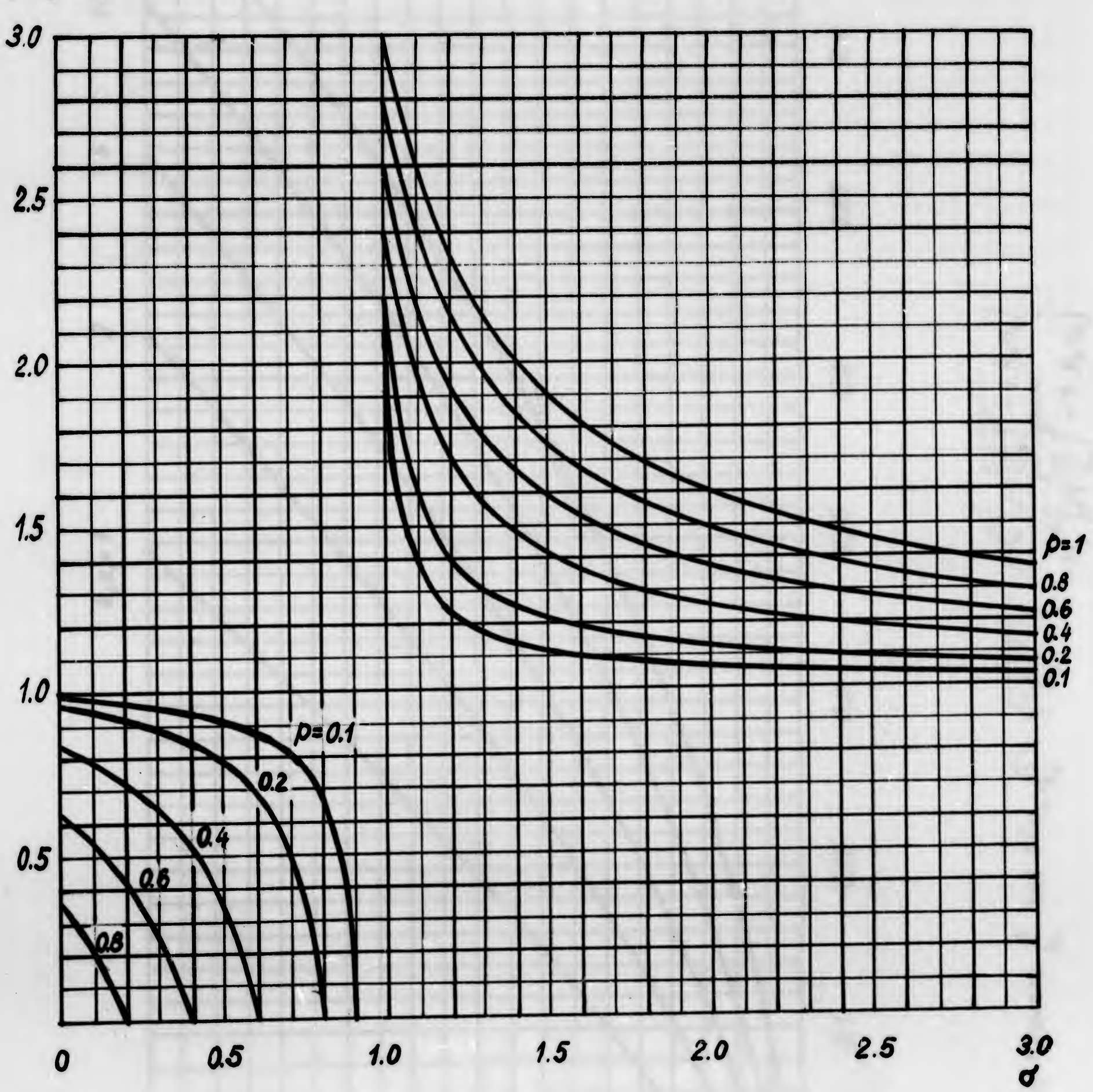

Fig. 2. $\nu(1-\varsigma^2)$ as a function of σ.

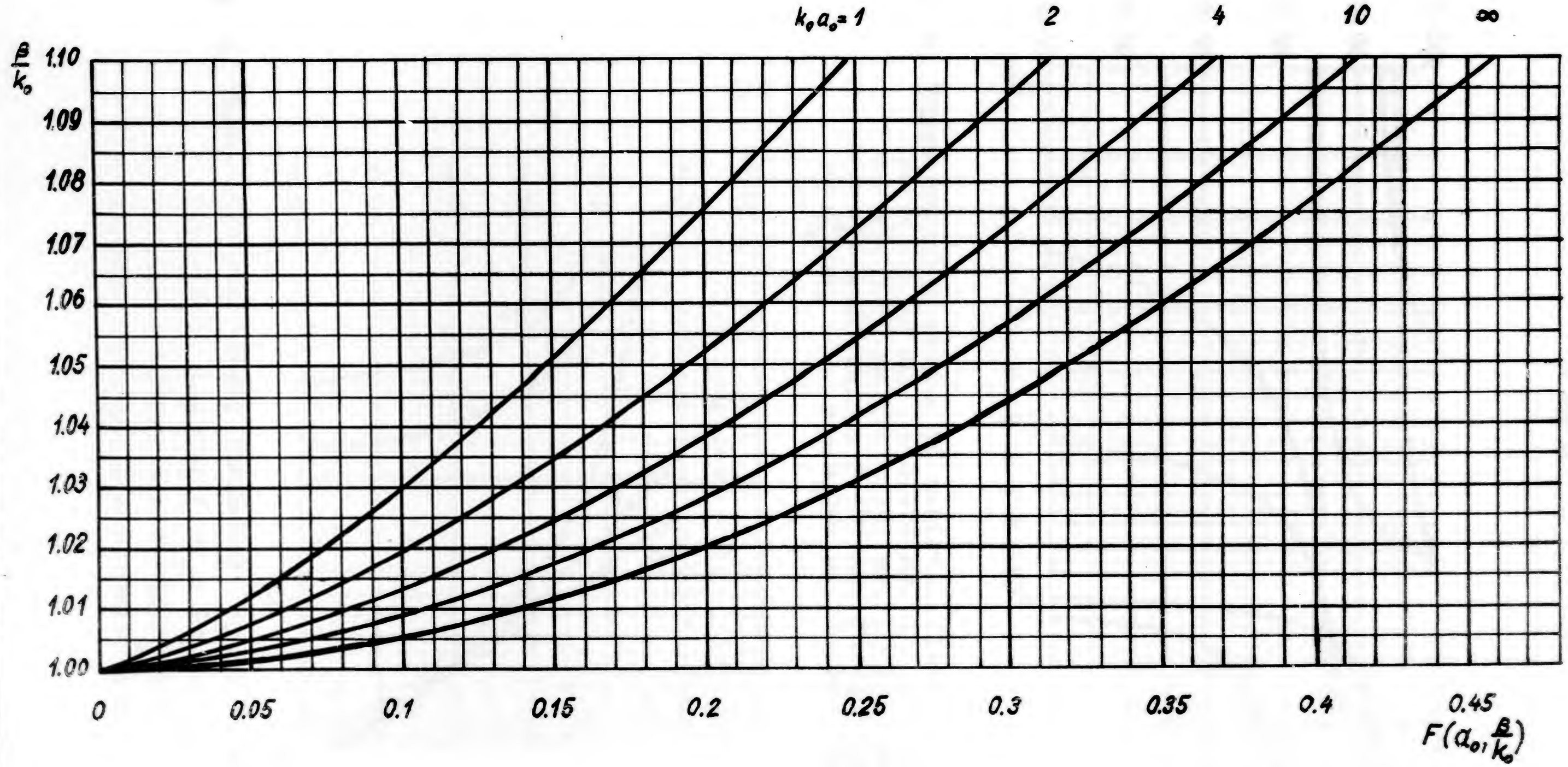

Fig. 3. Relationship between $\frac{\beta}{k_0}$ and $F(a_0, \frac{\beta}{k_0}) = \sqrt{(\frac{\beta}{k_0})^2 - 1}\ \frac{jH_0^{(1)}\left[j\sqrt{(\frac{\beta}{k_0})^2 - 1}\, k_0 a_0\right]}{-H_1^{(1)}\left[j\sqrt{(\frac{\beta}{k_0})^2 - 1}\, k_0 a_0\right]}$

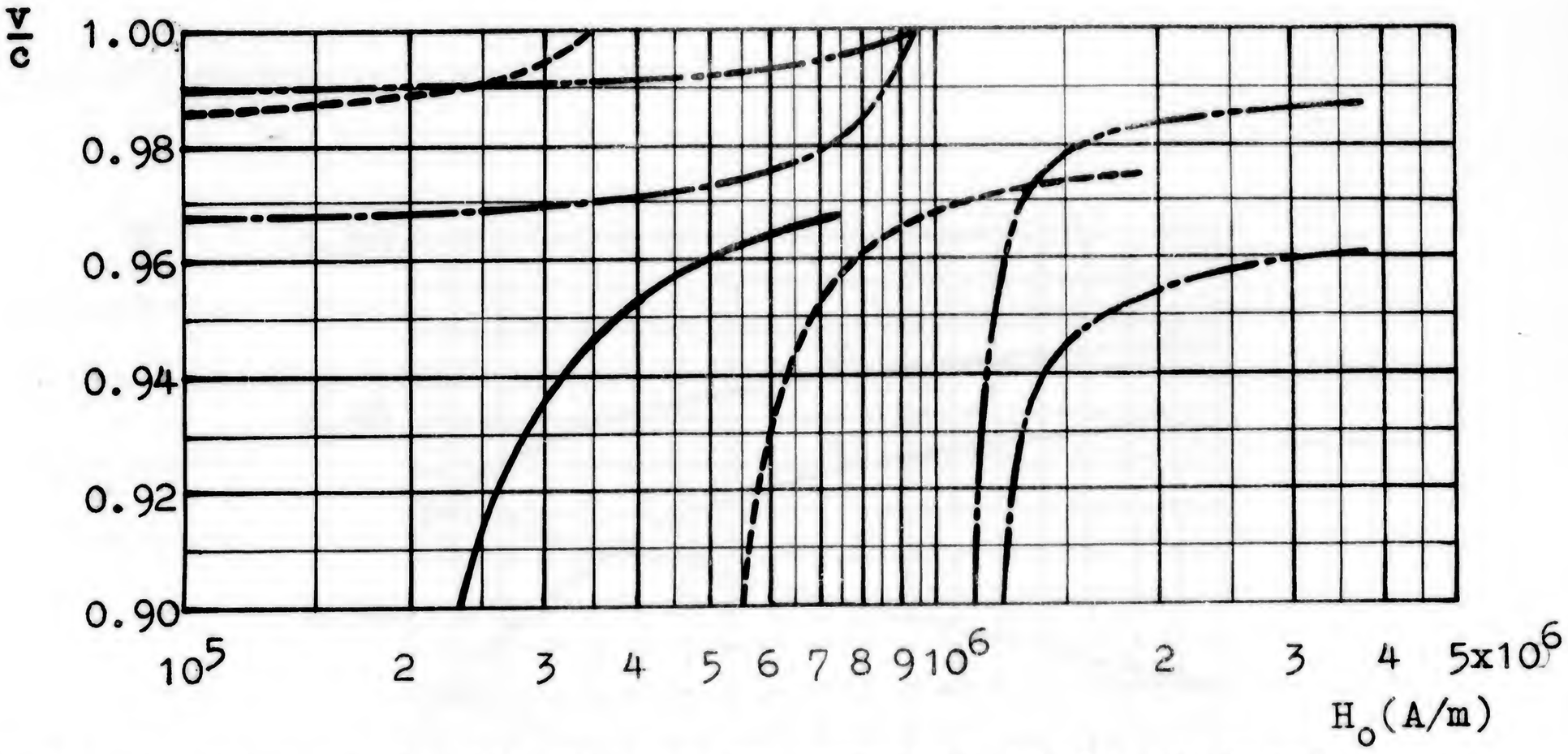

Fig. 4. Phase velocity versus DC magnetic field in ferrite.
—— a_o = 5.70 mm, δ = 0.478 mm, f = 8.4 kMc/s.
--- a_o = 2.28 mm, δ = 0.191 mm, f = 20.1 kMc/s.
-·- a_o = 2.28 mm, δ = 0.191 mm, f = 42.1 kMc/s.
-∞- a_o = ∞, δ = 0.191 mm, f = 42.1 kMc/s.

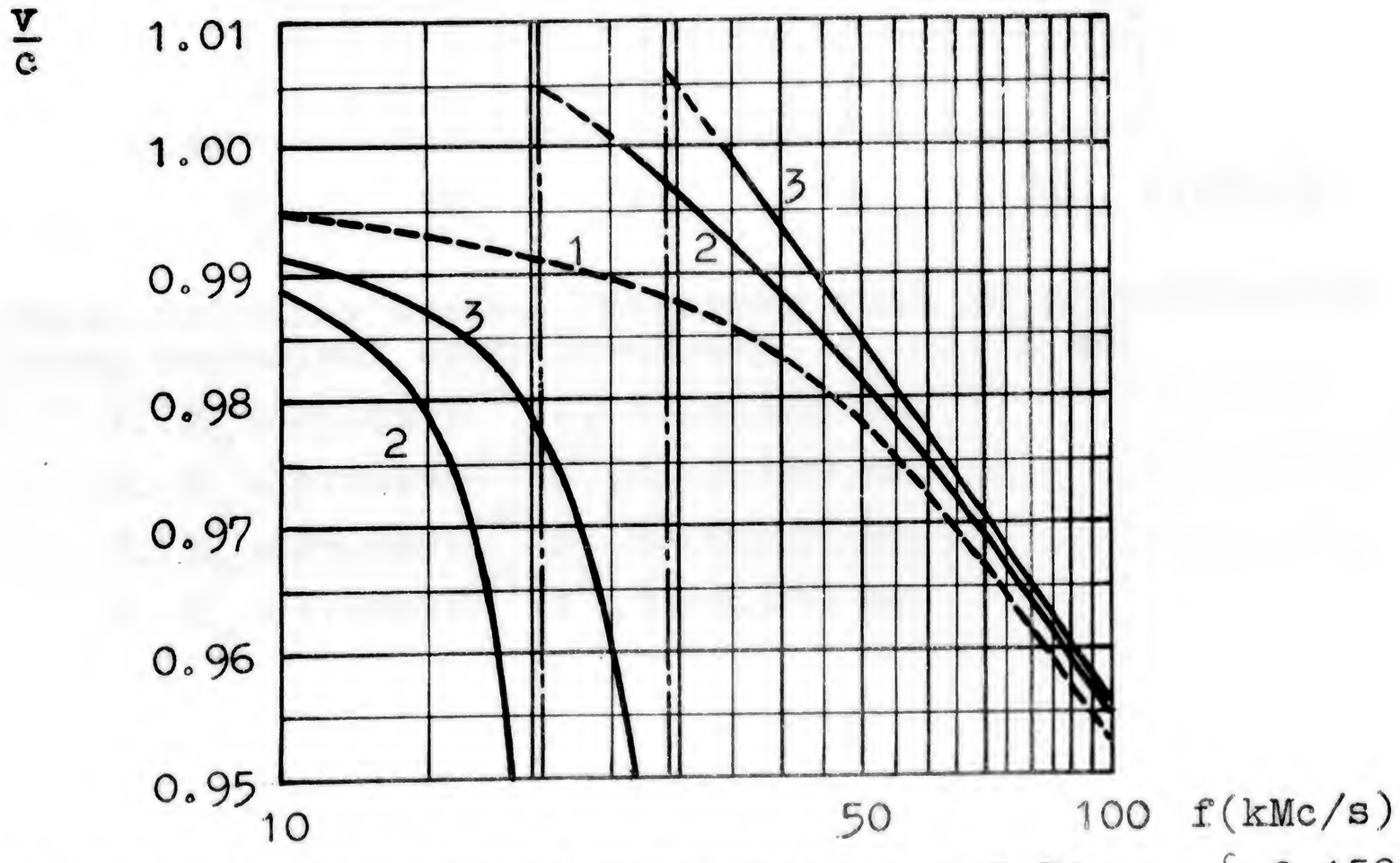

Fig. 5. Phase velocity versus frequency. a_o = 5.70 mm, δ=0.159 mm.
1. Dielectric coating ($\varepsilon/\varepsilon_o$ = 10, $\mu = \mu_o$).
2. Ferrite coating. H_o = 4.78x10^5 A/m.
3. Ferrite coating. H_o = 7.17x10^5 A/m.

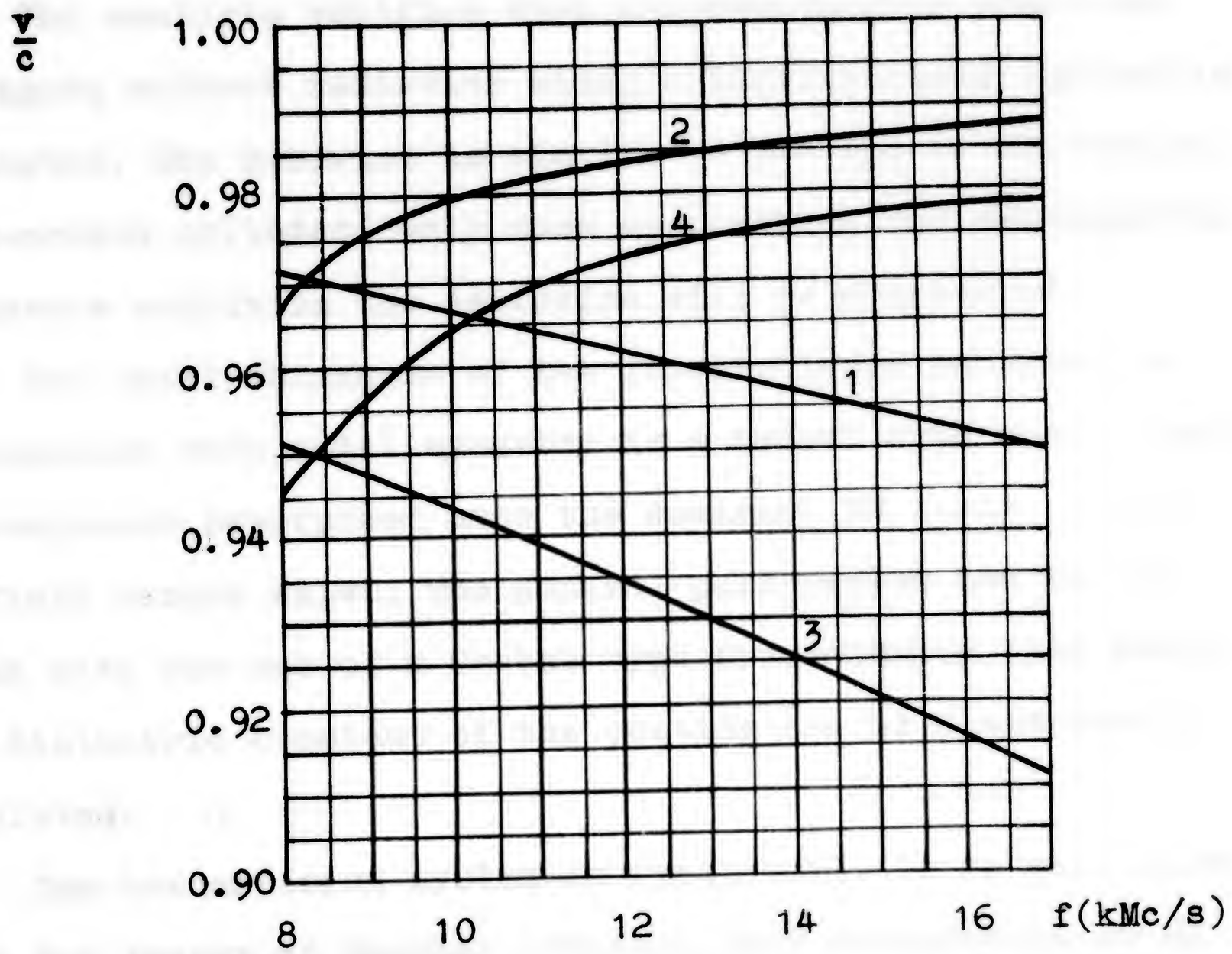

Fig.6. Phase velocity versus frequency with DC magnetization being dependent upon frequency. $a_o = 5.70$ mm.

1. $H_o = 25.6 \times 10^{-6} f$, $\delta = 0.159$ mm.
2. $H_o = 3.04 \times 10^{-15} f^2$, $\delta = 0.159$ mm.
3. $H_o = 25.6 \times 10^{-6} f$, $\delta = 0.239$ mm.
4. $H_o = 3.04 \times 10^{-15} f^2$, $\delta = 0.239$ mm.

7. CONCLUSION.

The analysis verifies that electromagnetic waves can propagate without radiation along a ferrite-coated cylindric conductor. The behavior is similar to the one of the dielectric-coated cylinder, only when approaching the gyromagnetic resonance condition the deviation will be pronounced.

For small thickness of the ferrite layer the mode of propagation with axial symmetry is a hybrid type with a small TE component superposed onto the dominant TM field. A pure TM field cannot exist. The general performance can be compared with the one of a Goubau type transmission line where the dielectric constant of the coating can be electrically regulated.

The transmission system is reciprocal. It is well known from the theory of Faraday rotation that propagation of an electromagnetic wave in the direction of the steady magnetic field in a ferrite material is non-reciprocal. Before the introduction of the ferrite-coating in the considered system this is reciprocal. The effect of the perturbation caused by the coating is small as long as this is sufficiently thin. This explains why the transmission system is reciprocal to the order of approximation here used.

As a possible application one may think of a phase shifter in an open transmission system like the Goubau line. Another possibility is the application of a finite length of

the transmission line as a surface wave antenna[9]. The radiation pattern of this may be adjusted by varying the propagation constant, which again can be done electrically through variation of the DC magnetization. This may be useful for tuning of antennas used over a wide frequency range or for splitting of the radiation pattern. The magnetization of the ferrite can also be made a function of the frequency so that the $\frac{v}{c}$ curve may obtain a rising character over a larger frequency range. This property usually leads to a greater constancy in the radiation pattern of an antenna, when the operating frequency is varied.

It should be mentioned that the radiation pattern of the antenna, due to the axial symmetry of the fields, will have a null in the axial direction. This type of antenna has been suggested used for certain kinds of direction finding equipment or as beacon antennas[10].

A more detailed discussion of the possible applications of the system treated in this paper must of course be postponed until a thorough analysis of the radiation properties has been carried out.

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APPENDIX I.

Review of Derivation of Fields with z-Dependence $e^{-j\beta z}$ in Unbounded Ferrite Medium. DC Magnetic Field in z-Direction.

It is convenient to introduce two parameters σ and p by the definitions

$$\sigma = \frac{|\gamma| H_o}{\omega} \quad \text{and} \quad p = \frac{|\gamma| M_o}{\mu_o \omega} \tag{18}$$

where $|\gamma|$ is the gyromagnetic ratio ($|\gamma| = 0.221 \frac{\text{M rad/s}}{\text{A/m}}$). H_o is the internal steady magnetic field and M_o is the saturation magnetization. For the lossless situation following expressions can be derived, as first done by Polder,

$$\nu = \frac{\mu}{\mu_o} = \frac{1 - p\sigma - \sigma^2}{1 - \sigma^2} \tag{19}$$

$$\varsigma = \frac{\kappa}{\mu} = \frac{p}{1 - p\sigma - \sigma^2} \tag{20}$$

Further a "starring operation" will be introduced by the definition

$$\bar{A} = (a,b)$$

$$\bar{A}^* = (b,-a)$$

Finally $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi})$.

With this notation Maxwell's equations for the ferrite medium can be written in the following forms

$$\nabla^* H_z + j\beta \bar{H}_t^* = j\omega\varepsilon \bar{E}_t \tag{21a}$$

$$\nabla \cdot \bar{H}_t^* = j\omega\varepsilon E_z \tag{21b}$$

$$\nabla^* E_z + j\beta \bar{E}_t^* = -j\omega\mu \bar{H}_t - \omega\varkappa \bar{H}_t^* \tag{22a}$$

$$\nabla \cdot \bar{E}_t^* = -j\omega\mu_o H_z \tag{22b}$$

where $\bar{H} = \bar{H}_t + H_z\hat{z}$ and $\bar{E} = \bar{E}_t + E_z\hat{z}$.

It is possible to simplify above set of equations by defining a certain normalizing convention. As a unit of length will be used $\frac{1}{\omega\sqrt{\mu_o\varepsilon}}$. The relation between the actual and the normalized propagation constants β_a and β, respectively, will then be

$$\beta_a = \beta\omega\sqrt{\mu_o\varepsilon} \tag{23a}$$

Further, $\bar{H}$ will be measured relatively to $\sqrt{\frac{\varepsilon}{\mu_o}}$ so that the normalized $\bar{H}$ has the dimension of an electric field.

$$\bar{H}_a = \bar{H}\sqrt{\frac{\varepsilon}{\mu_o}} \tag{23b}$$

In the normalized notation Maxwell's equations can be rewritten into

$$\nabla^* H_z + j\beta \bar{H}_t^* = j\bar{E}_t \tag{24a}$$

$$\nabla \cdot \bar{H}_t^* = jE_z \tag{24b}$$

$$\nabla^* E_z + j\beta \bar{E}_t^* = -\nu(j\bar{H}_t + \varsigma \bar{H}_t^*) \tag{25a}$$

$$\nabla \cdot \bar{E}_t^* = -jH_z \tag{25b}$$

where ν and ς have been substituted for $\frac{\mu}{\mu_o}$ and $\frac{\varkappa}{\mu}$, respectively. Elimination of $\bar{E}_t$ and $\bar{H}_t$ yields two simultaneous equations in the variables E_z and H_z. Below set of equations is obtained by following differential operations on equations (24a) and (25a): $\nabla \cdot$(24a), $\nabla \cdot$(25a), $\nabla^* \cdot$(24a) and $\nabla^* \cdot$(25a). For short $1 - \frac{\beta^2}{\nu}$ is denoted by q.

$$j\beta \nabla \cdot \bar{H}_t^* = j\nabla \cdot \bar{E}_t = -\beta E_z \tag{26a}$$

$$j\beta \nabla \cdot \bar{E}_t^* = -j\nu(\nabla \cdot \bar{H}_t + \varsigma E_z) = \beta H_z \tag{26b}$$

$$\nabla^2 H_z + j\beta \nabla \cdot \bar{H}_t = -j\nabla \cdot \bar{E}_t^* = -H_z \tag{27a}$$

$$\nabla^2 E_z + j\beta \nabla \cdot \bar{E}_t = -\nu(-j\nabla \cdot \bar{H}_t^* + \varsigma \nabla \cdot \bar{H}_t) = -\nu(E_z + \varsigma \nabla \cdot \bar{H}_t) \tag{27b}$$

Hence

$$\nabla^2 H_z + qH_z = j\beta\varsigma E_z \tag{28a}$$

$$\nabla^2 E_z + \nu(q - \varsigma^2)E_z = -j\beta\varsigma H_z \tag{28b}$$

It is evident from these equations that pure TE and TM fields do not exist. It will therefore be practical to determine the linear combinations of E_z and H_z that satisfies the simultaneous equations (28a) and (28b). Let such linear combinations be written as

(29) $$\psi = E_z + j\Lambda H_z$$

Multiplication of (28a) with $j\Lambda$ and adding to (28b) yields following equation

(30) $$\nabla^2\psi + (\nu(q-\varrho^2) + \beta\varrho\Lambda)E_z + j\Lambda(q + \frac{\beta\varrho}{\Lambda}) = 0$$

This is an equation of ψ alone of the form

(31) $$\nabla^2\psi + \chi^2\psi = 0$$

if and only if

$$\chi^2 = \nu(q-\varrho^2) + \beta\varrho\Lambda = q + \frac{\beta\varrho}{\Lambda}$$

or

(32) $$\Lambda^2 - \frac{q - \nu(q-\varrho^2)}{\beta\varrho}\Lambda - 1 = 0$$

If the roots of this equation are denoted by Λ_1 and Λ_2, respectively, it is seen that $\Lambda_1\Lambda_2 = -1$, and the corresponding values of χ^2 are

(33) $$\chi^2_{1,2} = \nu(q-\varrho^2) + \beta\varrho\Lambda_{1,2} = q - \beta\varrho\Lambda_{2,1}$$

Introducing the substitution $\lambda = \beta\Lambda$ equation (32) can be written

(34) $$\lambda^2 - \frac{q - \nu(q-\varrho^2)}{\varrho}\lambda - \beta^2 = 0$$

Utilizing (19), (20), (33) and (34) $\chi_{1,2}$ can be expressed exclusively in terms of $\lambda_{1,2}$ and σ

$$\chi^2_{1,2} = \frac{1-\lambda^2_{1,2}}{1-\sigma\lambda_{1,2}} \tag{35}$$

and (34) becomes

$$\lambda^2 - (p+\sigma(1-\beta^2))\lambda - \beta^2 = 0 \tag{36}$$

From (29) is seen that

$$\psi_1 = E_z + j\Lambda_1 H_z$$

$$\psi_2 = E_z + j\Lambda_2 H_z$$

Hence

$$E_z = \frac{\Lambda_2\psi_1 - \Lambda_1\psi_2}{\Lambda_2 - \Lambda_1} \tag{37a}$$

$$H_z = j\frac{\psi_1 - \psi_2}{\Lambda_2 - \Lambda_1} \tag{37b}$$

To satisfy the boundary conditions in the particular problem investigated the φ-variation of the solution for the fields must be given by $e^{jn\varphi}$, where n is any integer. Assuming this φ-dependence (31) can be written

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi_{1,2}}{\partial r}) + (\chi^2_{1,2} - \frac{n^2}{r^2})\psi_{1,2} = 0 \tag{38}$$

These equations have the solutions

$$\Psi_{1,2} = A'_{1,2} H_n^{(1)}(\chi_{1,2} r) + B'_{1,2} H_n^{(2)}(\chi_{1,2} r) \tag{39}$$

where the factor $e^{jn\varphi}$ is omitted for simplicity. The four constants A'_1, A'_2, B'_1 and B'_2 can be determined by the boundary conditions of the specific problem.

It is convenient to denote the starring operation by the symbol P. With this notation equations (24a) and (25a) can be rewritten into

$$jP\bar{E}_t + j\beta\bar{H}_t = -\nabla H_z \tag{40a}$$

$$-j\beta\bar{E}_t + \nu(jP - \varrho)\bar{H}_t = \nu E_z \tag{40b}$$

The solutions with respect to $\bar{E}_t$ and $\bar{H}_t$ are

$$\nu(q - j\varrho P)\bar{E}_t = -j\beta\nu E_z - \nu(jP - \varrho)\nabla H_z \tag{41a}$$

$$\nu(q - j\varrho P)\bar{H}_t = j(P\nu E_z - \beta\nu H_z) \tag{41b}$$

Using the identity $a - P = \dfrac{1 + a^2}{a + P}$

$$\nu(q - j\varrho P) = j\varrho\nu\left(\frac{q}{j\varrho} - P\right) = j\varrho\nu \frac{1 - \dfrac{q^2}{\varrho^2}}{\dfrac{q}{j\varrho} + P} = \frac{\nu(q^2 - \varrho^2)}{q + j\varrho P}$$

Insertion of this expression into (41a) and (41b) yields

$$\nu(q^2 - \varrho^2)\bar{E}_t = (q + j\varrho P)(-j\beta\nu E_z - \nu(jP - \varrho)\nabla H_z)$$

$$\nu(q^2-\varrho^2)\bar{H}_t = (jq-\varrho P)(P\nabla E_z - \beta\nabla H_z)$$

Returning to the star notation instead of using the symbol P and further applying equations (37a) and (37b) these equations may be written in the final form

(42a) $$\nu(q^2-\varrho^2)(\Lambda_1-\Lambda_2)\bar{E}_t = j\beta(q(\Lambda_2\nabla\psi_1 - \Lambda_1\nabla\psi_2) + \beta\varrho(\nabla\psi_1 - \nabla\psi_2))$$
$$-(\beta\varrho(\Lambda_2\nabla^*\psi_1 - \Lambda_1\nabla^*\psi_2) + \nu(q-\varrho^2)(\nabla^*\psi_1 - \nabla^*\psi_2))$$

(42b) $$\nu(q^2-\varrho^2)(\Lambda_1-\Lambda_2)\bar{H}_t = -\varrho(\Lambda_2\nabla\psi_1 - \Lambda_1\nabla\psi_2) - \beta q(\nabla\psi_1 - \nabla\psi_2)$$
$$-j(q(\Lambda_2\nabla^*\psi_1 - \Lambda_1\nabla^*\psi_2) + \beta\varrho(\nabla^*\psi_1 - \nabla^*\psi_2))$$

APPENDIX II.

Equations (1a), (1b), (3a), (3b), (6a) and (6c) yield

(43a) $$(\Lambda_2-\Lambda_1)\left[E_z\right]_{r=a+\delta} = \left[\Lambda_2\psi_1 - \Lambda_1\psi_2\right]_{r=a+\delta} = -(\Lambda_1-\Lambda_2)Cp$$

(43b) $$-j(\Lambda_2-\Lambda_1)\left[H_z\right]_{r=a+\delta} = \left[\psi_1-\psi_2\right]_{r=a+\delta} = j(\Lambda_1-\Lambda_2)Dp$$

Here and in the following $r = a+\delta$ is inserted in the expressions for p and p′.

By above equations and (8a) and (8b)

(44a) $$\sqrt{a+\delta}\left[\psi_1\right]_{r=a+\delta} = A_1 e^{j\chi_1(a+\delta)} + B_1 e^{-j\chi_1(a+\delta)} = p\sqrt{a+\delta}(C+j\Lambda_1 D)$$

(44b) $\sqrt{a+\delta}\,[\psi_2]_{r=a+\delta} = A_2 e^{j\chi_2(a+\delta)} + B_2 e^{-j\chi_2(a+\delta)} = p\sqrt{a+\delta}\,(C + j\Lambda_2 D)$

Utilizing the assumption $|\chi|\delta \ll 1$ and introducing the parameters t_1, u_1, t_2 and u_2 defined by

$$
\begin{aligned}
t_1 &= A_1 e^{j\chi_1 a} + B_1 e^{-j\chi_1 a} = [\sqrt{r}\,\psi_1]_{r=a} \\
u_1 &= A_1 e^{j\chi_1 a} - B_1 e^{-j\chi_1 a} = \frac{1}{j\chi_1}\left[\frac{\partial}{\partial r}(\sqrt{r}\,\psi_1)\right]_{r=a} \\
t_2 &= A_2 e^{j\chi_2 a} + B_2 e^{-j\chi_2 a} = [\sqrt{r}\,\psi_2]_{r=a} \\
u_2 &= A_2 e^{j\chi_2 a} - B_2 e^{-j\chi_2 a} = \frac{1}{j\chi_2}\left[\frac{\partial}{\partial r}(\sqrt{r}\,\psi_2)\right]_{r=a}
\end{aligned}
\tag{45}
$$

equations (44a) and (44b) may be rewritten into

(46a) $\sqrt{a+\delta}\,[\psi_1]_{r=a+\delta} = (1+j\chi_1\delta)A_1 e^{j\chi_1 a} + (1-j\chi_1\delta)B_1 e^{-j\chi_1 a}$

$= t_1 + j\chi_1\delta u_1 = p\sqrt{a+\delta}\,(C + j\Lambda_1 D)$

(46b) $\sqrt{a+\delta}\,[\psi_2]_{r=a+\delta} = (1+j\chi_2\delta)A_2 e^{j\chi_2 a} + (1-j\chi_2\delta)B_2 e^{-j\chi_2 a}$

$= t_2 + j\chi_2\delta u_2 = p\sqrt{a+\delta}\,(C + j\Lambda_2 D)$

From equations (3a), (5a), (8a), (8b) and (45) is seen

(47) $\sqrt{a}\,[\Lambda_2\psi_1 - \Lambda_1\psi_2]_{r=a} = \Lambda_2 t_1 - \Lambda_1 t_2 = 0$

The relation

(48) $\frac{\partial}{\partial r}\psi = \frac{1}{\sqrt{r}}\frac{\partial}{\partial r}(\sqrt{r}\,\psi) - \frac{1}{2r}\psi$

will be frequently used in the following.

From equation (3c) it is seen that (5b) is fulfilled if

$$(49)\quad \nu(q^2-\varrho^2)(\Lambda_1-\Lambda_2)\left[E_\varphi\right]_{r=a} = -\frac{n\beta}{a}\left[q(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta\varrho(\psi_1-\psi_2)\right]_{r=a} + \left[\frac{\partial}{\partial r}(\beta\varrho(\Lambda_2\psi_1-\Lambda_1\psi_2)+\nu(q-\varrho^2)(\psi_1-\psi_2))\right]_{r=a} = 0$$

By use of (45), (47), (48) equation (49) can be written

$$(50)\quad -\frac{1}{a}\left(\frac{\nu(q-\varrho^2)}{2}+n\beta^2\varrho\right)(t_1-t_2)+j\beta\varrho(\Lambda_2\chi_1u_1-\Lambda_1\chi_2u_2) + j\nu(q-\varrho^2)(\chi_1u_1-\chi_2u_2) = 0$$

Now u_1 and u_2 can be eliminated. Equations (46a), (46b) and (47) yield

$$(51a)\quad j(\Lambda_2\chi_1u_1-\Lambda_1\chi_2u_2)\delta = p\sqrt{a+\delta}(\Lambda_2-\Lambda_1)C$$

$$(51b)\quad j(\chi_1u_1-\chi_2u_2)\delta = jp\sqrt{a+\delta}(\Lambda_1-\Lambda_2)D-(t_1-t_2)$$

Insertion of this into (50) gives following expression for t_1-t_2

$$(52)\quad t_1-t_2 = p\sqrt{a+\delta}(\Lambda_1-\Lambda_2)\,\frac{-\beta\varrho C+j\nu(q-\varrho^2)D}{\nu(q-\varrho^2)(1+\frac{\delta}{2a})+n\beta^2\varrho\frac{\delta}{a}}$$

From equations (8a), (8b) and (45) is seen that

$$(53a)\quad \left[\frac{\partial}{\partial r}(\sqrt{r}\,\psi_1)\right]_{r=a+\delta} = j\chi_1(u_1+j\chi_1\delta t_1)$$

(53b) $$\left[\frac{\partial}{\delta r}(\sqrt{r}\,\Psi_2)\right]_{r=a+\delta} = jx_2(u_2 + jx_2\delta t_2)$$

These expressions and formulas (33) and (47) justify following rewritings

(54a) $$\left[\frac{\partial}{\delta r}(\sqrt{r}(\Lambda_2\Psi_1-\Lambda_1\Psi_2))\right]_{r=a+\delta}$$

$$= j(\Lambda_2 x_1 u_1-\Lambda_1 x_2 u_2) - \delta(\Lambda_2 x_1^2 t_1-\Lambda_1 x_2^2 t_2)$$

$$= -\frac{1}{\delta}\, p\sqrt{a+\delta}\,(\Lambda_1-\Lambda_2)C + \beta\varsigma\delta(t_1-t_2)$$

(54b) $$\left[\frac{\partial}{\delta r}(\sqrt{r}(\Psi_1-\Psi_2))\right]_{r=a+\delta} = j(x_1u_1-x_2u_2) - \delta(x_1^2t_1-x_2^2t_2)$$

$$= j\frac{1}{\delta}\, p\sqrt{a+\delta}\,(\Lambda_1-\Lambda_2)D - (\frac{1}{\delta}+\delta q)(t_1-t_2)$$

Equations (3c), (43a), (43b), (48), (52), (54a) and (54b) together with the definition of q yield

(55) $$\nu(q^2-\varsigma^2)(\Lambda_1-\Lambda_2)\left[E_\varphi\right]_{r=a+\delta} = -\frac{n\beta}{a+\delta}\left[q(\Lambda_2\Psi_1-\Lambda_1\Psi_2) + \beta\varsigma(\Psi_1-\Psi_2)\right]_{r=a+\delta}$$

$$+ \left[\frac{\partial}{\delta r}(\beta\varsigma(\Lambda_2\Psi_1-\Lambda_1\Psi_2)+\nu(q-\varsigma^2)(\Psi_1-\Psi_2)\right]_{r=a+\delta}$$

$$= -\frac{1}{a+\delta}\left[\beta(\frac{\varsigma}{2}+nq)(\Lambda_2\Psi_1-\Lambda_1\Psi_2)+(\frac{\nu(q-\varsigma^2)}{2}+n\beta^2\varsigma)(\Psi_1-\Psi_2)\right]_{r=a+\delta}$$

$$+\frac{1}{\sqrt{a+\delta}}\left[\frac{\partial}{\delta r}(\sqrt{r}(\beta\varsigma(\Lambda_2\Psi_1-\Lambda_1\Psi_2)+\nu(q-\varsigma^2)(\Psi_1-\Psi_2))\right]_{r=a+\delta}$$

$$= \frac{p(\Lambda_1-\Lambda_2)}{a+\delta}\left[\beta(\frac{\varsigma}{2}+nq)C - j(\frac{\nu(q-\varsigma^2)}{2}+n\beta^2\varsigma)D\right]$$

$$+ p(\Lambda_1-\Lambda_2)(-\beta\varsigma C+j\nu(q-\varsigma^2)D)\frac{\frac{1}{a}(\frac{\nu(q-\varsigma^2)}{2}+n\beta^2\varsigma)-\nu(q^2-\varsigma^2)\delta}{\nu(q-\varsigma^2)(1+\frac{\delta}{2a})+n\beta^2\varsigma\frac{\delta}{a}}$$

$$= \frac{p(\Lambda_1-\Lambda_2)}{a+\delta}(q^2-\varsigma^2)\left[\beta\frac{\varsigma\delta(a+\delta)-\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}+n(1+\frac{\delta}{2a})+n^2q\frac{1-q}{q^2-\varsigma^2}\varsigma\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}C\right.$$

$$\left.-j\nu\frac{(q-\varsigma^2)(\delta(a+\delta)-\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a})+n^2\frac{(1-q)^2}{q^2-\varsigma^2}\varsigma^2\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}D\right]$$

By equations (1d), (6b) and (55) following characteristic equation is obtained

$$(56)\quad \beta\left[\frac{\varsigma\delta(a+\delta)-\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}+n(1+\frac{\delta}{2a})+n^2\frac{1-q}{q^2-\varsigma^2}q\varsigma\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}+jn\nu\frac{1}{k_1^2}\right]C$$

$$-j\nu\left[\frac{(q-\varsigma^2)\left[\delta(a+\delta)-\frac{q-\varsigma^2}{q^2-\varsigma^2}\frac{\delta}{4a}\right]+n^2\frac{(1-q)^2}{q^2-\varsigma^2}\varsigma^2\frac{\delta}{a}}{(q-\varsigma^2)(1+\frac{\delta}{2a})+n(1-q)\varsigma\frac{\delta}{a}}+\frac{1}{k_1}\frac{p'}{p}(a+\delta)\right]D=0$$

From the same equations as previously, except for the application of (3d) instead of (3c), below derivations can be performed

$$(57)\quad j\nu(q^2-\varsigma^2)(\Lambda_1-\Lambda_2)\left[H_\varphi\right]_{r=a+\delta}=\frac{n}{a+\delta}\left[\varsigma(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta q(\psi_1-\psi_2)\right]_{r=a+\delta}$$

$$-\left[\frac{\partial}{\partial r}\left(q(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta\varsigma(\psi_1-\psi_2)\right)\right]_{r=a+\delta}$$

$$=\frac{1}{a+\delta}\left[\left(\frac{q}{2}+n\varsigma\right)(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta\left(\frac{\varsigma}{2}+nq\right)(\psi_1-\psi_2)\right]_{r=a+\delta}$$

$$-\frac{1}{\sqrt{a+\delta}}\left[\frac{\varsigma}{\delta r}\left(\sqrt{r}\left[q(\Lambda_2\psi_1-\Lambda_1\psi_2)+\beta\varsigma(\psi_1-\psi_2)\right]\right)\right]_{r=a+\delta}$$

$$=\frac{p(\Lambda_1-\Lambda_2)}{a+\delta}\left(-\left(\frac{q}{2}+n\varsigma\right)C+j\beta\left(\frac{\varsigma}{2}+nq\right)D\right)$$

$$+\frac{1}{\delta}\,p(\Lambda_1-\Lambda_2)\frac{\left[q^2-\varsigma^2+q(q-\varsigma^2)\frac{\delta}{2a}+nq(1-q)\varsigma\frac{\delta}{a}\right]C-j\beta\varsigma\left[\frac{q-\varsigma^2}{2}+n(1-q)\right]\frac{\delta}{a}\,D}{(q-\varsigma^2)\left(1+\frac{\delta}{2a}\right)+n(1-q)\varsigma\frac{\delta}{a}}$$

$$=\frac{p(\Lambda_1-\Lambda_2)}{a+\delta}(q^2-\varsigma^2)\left[\frac{\frac{1}{\delta}(a+\delta)+q\,\frac{q-\varsigma^2}{q^2-\varsigma^2}\,\frac{\delta}{4a}-n\varsigma\left(1+\frac{\delta}{2a}\right)-n^2\,\frac{(1-q)\varsigma^2}{q^2-\varsigma^2}\,\frac{\delta}{a}}{(q-\varsigma^2)\left(1+\frac{\delta}{2a}\right)+n(1-q)\varsigma\,\frac{\delta}{a}}\,C\right.$$

$$\left.-j\beta\,\frac{\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\,\frac{\delta}{4a}-n\left(1+\frac{\delta}{2a}\right)-n^2q(1-q)\varsigma\frac{\delta}{a}}{(q-\varsigma^2)\left(1+\frac{\delta}{2a}\right)+n(1-q)\varsigma\,\frac{\delta}{a}}\,D\right]$$

Finally, by equations (1f), (6d) and (57) a second characteristic equation is obtained, which together with (56) determines the constants of the surface wave.

$$(58)\quad \left[\frac{\frac{1}{\delta}(a+\delta)+q\,\frac{q-\varrho^2}{q^2-\varsigma^2}\,\frac{\delta}{4a}-n\varsigma\left(1+\frac{\delta}{2a}\right)-n^2\,\frac{(1-q)\varrho^2}{q^2-\varrho^2}\,\frac{\delta}{a}}{(q-\varsigma^2)\left(1+\frac{\delta}{2a}\right)+n(1-q)\varsigma\,\frac{\delta}{a}}-\frac{\nu}{k_1}\,\frac{\varepsilon_0}{\varepsilon}\,\frac{p'}{p}(a+\delta)\right]C$$

$$-j\beta\left[\frac{\varsigma\frac{q-\varsigma^2}{q^2-\varsigma^2}\,\frac{\delta}{4a}-n\left(1+\frac{\delta}{2a}\right)-n^2q(1-q)\varsigma\frac{\delta}{a}}{(q-\varsigma^2)\left(1+\frac{\delta}{2a}\right)+n(1-q)\varsigma\,\frac{\delta}{a}}-n\nu\frac{1}{k_1^2}\right]D=0$$

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